

A NATURAL DUALITY FOR INVOLUTIVE BISEMILATTICES

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ABSTRACT. We establish a natural duality between the category of involutive bisemilattices and the category of strongly inverse systems of Stone spaces, using Stone duality from one side and the representation of involutive bisemilattices as Plonka sum of Boolean algebras, from the other. Furthermore, we show that the dual space of an involutive bisemilattice can be viewed as a GR space with involution, a generalization of the spaces introduced by Gierz and Romanowska [2] equipped with an involution as additional operation.

1. INTRODUCTION

It is a common trend in mathematics to study natural dualities for general algebraic structures and, in particular, for those arising from mathematical logic. The first step towards this direction traces back to the pioneering work by Stone for Boolean algebras [16]. Later on, Stone duality has been extended to the more general case of distributive lattices by Priestley [12]. The two above mentioned are the prototypical examples of natural dualities and will be both recalled and constructively used in the present work.

A natural duality has an intrinsic value: it is a way of describing the very same mathematical object from two different perspectives, the target category and its dual.

The starting point of our analysis is the duality established by Gierz and Romanowska [2] between distributive bisemilattices and compact totally disconnected partially ordered left normal bands with constants, which we refer to as GR spaces. The relevance of the result lies mainly in using the technique of Plonka sums as an essential tool for proving the duality.

Our aim is to provide a natural duality between the categories of involutive bisemilattices and those topological spaces, here christened as GR spaces with involution. The former consists of a class of algebras introduced and extensively studied in [1] as algebraic semantics (although not equivalent) for paraconsistent weak Kleene logic. Involutive bisemilattices are strictly connected to Boolean algebras as they are representable as Plonka sums of Boolean algebras.

The present work consists of two main results. On one hand, taking advantage of the Plonka sums representation in terms of Boolean algebras and Stone duality, we

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are able to describe the dual space of an involutive bisemilattice as strongly inverse systems of Stone spaces (Theorem 4.6). On the other hand, we generalize Gierz and Romanowska duality by considering GR spaces with involution as an additional operation (Theorem 4.18). As a byproduct of our analysis we get a topological description of *strongly inverse systems* of Stone spaces (Corollary 4.19).

The paper is structured as follows. In Section 2 we summarize all the necessary notions and known results about bisemilattices, Gierz and Romanowska duality and involutive bisemilattices. In Section 3 we the categories of *strongly* direct and inverse systems, proving that they are dual. In Section 4, we introduce GR spaces with involution and prove the main results. Finally, in Section 5 we make some considerations about categories admitting both topological duals and a representation in terms of Płonka sums. By using Priestley duality, we then extend our results to the category of distributive bisemilattices.

2. PRELIMINARIES

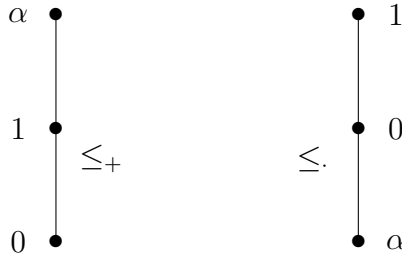
A *distributive bisemilattice* is an algebra $\mathbf{A} = \langle A, +, \cdot \rangle$ of type $\langle 2, 2 \rangle$ such that both $+$ and \cdot are idempotent, associative and commutative operations and, moreover, $+$ (\cdot respectively) distributes over \cdot ($+$ respectively). Distributive bisemilattices have been introduced by Płonka [9], who called them “quasi lattices”; nowadays these structures are studied in a more general setting under the name of Birkhoff systems (see [4], [5]). Throughout the paper we will refer to these algebras simply as *bisemilattices*. Observe that every distributive lattice is an example of bisemilattice and every semilattice is a bisemilattice, where the two operations coincide. Any bisemilattice induces two different partial orders, namely $x \leq y$ iff $x \cdot y = x$ and $x \leq_+ y$ iff $x + y = y$.

Example 2.1. The 3-element algebra $\mathbf{3} = \langle \{0, 1, \alpha\}, \cdot, + \rangle$, whose operations are defined by the so-called *weak Kleene tables*:

\cdot	0	α	1
0	0	α	0
α	α	α	α
1	0	α	1

$+$	0	α	1
0	0	α	1
α	α	α	α
1	1	α	1

The two partial orders may be represented by the following Hasse diagrams:



It has been proven in [6] that the variety of bisemilattices is generated by **3**.

A theory of natural duality for bisemilattices has been developed in [2]. We recall here all the notions needed to state the main result.

A *left normal band* is an idempotent semigroup $\langle A, * \rangle$ satisfying the additional identity $x * (y * z) = x * (z * y)$, which is weak form of commutativity. A left normal band can be equipped with a partial order.

Definition 2.2. A *partially ordered left normal band* is an algebra $\mathbf{A} = \langle A, *, \leq \rangle$ such that

- i) $\langle A, * \rangle$ is a left normal band
- ii) $\langle A, \leq \rangle$ is a partially ordered set
- iii) if $x \leq y$ then $x * z \leq y * z$ and $z * x \leq z * y$
- iv) $x * y \leq x$

In any partially ordered left normal band it is possible to define a second partial order via $*$ and \leq : $a \sqsubseteq b$ iff $a * b \leq b$ and $b * a = b$. A partially ordered left normal band may be equipped with constants.

Definition 2.3. A *partially ordered left normal band with constants* is an algebra $\mathbf{A} = \langle A, *, \leq, c_0, c_1, c_\alpha \rangle$ such that $\langle A, *, \leq \rangle$ is a partially ordered left normal band and c_0, c_1 and c_α are constants satisfying

- (1) $x * c_\alpha = c_\alpha * x = c_\alpha$
- (2) $x * c_0 = x * c_1 = x$
- (3) $c_0 \sqsubseteq x \leq c_1$ and $c_\alpha \leq x \sqsubseteq c_\alpha$
- (4) if $c_0 * x = c_1 * x$ then $x = c_\alpha$

Definition 2.4. A *GR space* is a structure $\mathbf{A} = \langle A, *, \leq, c_0, c_1, c_\alpha, \tau \rangle$, such that $\langle A, *, \leq, c_0, c_1, c_\alpha \rangle$ is a partially ordered left normal band with constants and τ is a topology making $*$: $A \times A \rightarrow A$ a continuous map and $\langle A, \leq, \tau \rangle$ is a *totally order disconnected space*¹.

Example 2.5. The support set of **3**, namely $\{0, 1, \alpha\}$ equipped with the discrete topology, where $\leq \equiv \leq_+$, $c_0 = 0$, $c_1 = 1$, $c_\alpha = \alpha$ and $*$ is defined as follows:

$$a * b = \begin{cases} a & \text{if } b \neq \alpha \\ b & \text{otherwise} \end{cases}$$

is a GR space (it is not difficult to check that operation $a * b = a + a \cdot b = a \cdot (a + b)$ and that the induced order \sqsubseteq coincides with \leq_+).

We call \mathfrak{B} the category of bisemilattices (whose morphisms are homomorphisms of bisemilattices) and \mathfrak{GR} the category of GR spaces (whose morphisms are continuous maps preserving $*$, constants and the order). The main result in [2] is the following:

¹A topological space is totally order disconnected if (1) $\{(a, b) \in A \times A : a \leq b\}$ is closed; (2) if $a \not\leq b$ then there is an open and closed (clopen) lower set U such that $b \in U$ and $a \notin U$.

Theorem 2.6. [2, Theorem 7.5] *The categories \mathfrak{DB} and \mathfrak{GR} are dual to each other under the invertible functor $\text{Hom}_{\mathfrak{b}}(-, \mathbf{3}) : \mathfrak{DB} \rightarrow \mathfrak{GR}$ and its inverse $\text{Hom}_{\mathfrak{GR}}(-, \mathbf{3}) : \mathfrak{GR} \rightarrow \mathfrak{DB}$.*

In detail, given a bisemilattice \mathbf{S} , its dual GR space is $\hat{\mathbf{S}} = \text{Hom}_{\mathfrak{b}}(\mathbf{S}, \mathbf{3})$, i.e. the space of the homomorphisms (of bisemilattices) from \mathbf{S} to $\mathbf{3}$. Analogously, if \mathbf{A} is a GR space, then the dual is given by $\hat{\mathbf{A}} = \text{Hom}_{\mathfrak{GR}}(\mathbf{A}, \mathbf{3})$, the bisemilattice of morphisms of \mathfrak{GR} .

The isomorphism between \mathbf{S} and $\hat{\hat{\mathbf{S}}}$ is given by:

$$\varepsilon_S : \mathbf{S} \rightarrow \hat{\hat{\mathbf{S}}}, x \mapsto \varepsilon_S(x), \varepsilon_S(x)(\varphi) = \varphi(x), \quad (1)$$

for every $x \in S$ and $\varphi \in \hat{S}$.

Analogously, for \mathbf{A} and $\hat{\hat{\mathbf{A}}}$, the isomorphism is given by:

$$\delta_A : \mathbf{A} \rightarrow \hat{\hat{\mathbf{A}}}, x \mapsto \varepsilon_A(x), \varepsilon_A(x)(\varphi) = \varphi(x), \quad (2)$$

for every $x \in A$ and $\varphi \in \hat{A}$.

The class of *involutive bisemilattices* has been introduced in [1] as the most suitable candidate to be the algebraic counterpart of PWK logic.

Definition 2.7. An *involutive bisemilattice* is an algebra $\mathbf{B} = \langle B, \cdot, +, ', 0, 1 \rangle$ of type $(2, 2, 1, 0, 0)$ satisfying:

- I1. $x + x = x$;
- I2. $x + y = y + x$;
- I3. $x + (y + z) = (x + y) + z$;
- I4. $(x')' = x$;
- I5. $x \cdot y = (x' + y')'$;
- I6. $x \cdot (x' + y) = x \cdot y$;
- I7. $0 + x = x$;
- I8. $1 = 0'$.

We denote the variety of involutive bisemilattices by \mathcal{IBSL} .

Every involutive bisemilattice has, in particular, the structure of a join semilattice with zero, in virtue of axioms (I1)–(I3) and (I7). More than that, it is possible to prove [1, Proposition 20] that \cdot distributes over $+$ and viceversa, therefore the reduct $\langle B, +, \cdot \rangle$ is a bisemilattice. Notice that, in virtue of axioms (I5) and (I8), the operations \cdot and 1 are completely determined by $+$, $'$, and 0 . It is not difficult to check that every involutive bisemilattice has also the structure of a meet semilattice with 1 , and that the equations $x + y = (x' \cdot y')'$, $x + y = x + (x' \cdot y)$ are satisfied. There are different equivalent ways to define involutive bisemilattices: it is not difficult to check that \mathcal{IBSL} corresponds to the regularization of the variety of Boolean algebras described in [11].

Example 2.8. Every Boolean algebra, in particular the 2-element Boolean algebra \mathbf{B}_2 , is an involutive bisemilattice. Also, the 2-element semilattice with zero, which we call \mathbf{S}_2 , endowed with identity as its unary fundamental operation, is an involutive bisemilattice. The most prominent example of involutive bisemilattice is the 3-element algebra \mathbf{WK} , which is obtained by expanding the language of $\mathbf{3}$ with an involution behaving as follows:

'	
1	0
α	α
0	1

Upon considering the partial order \leq induced by the product in its bisemilattice reduct, it becomes a 3-element chain with α as its bottom element.

It is not difficult to verify that \mathbf{B}_2 is a subalgebra of \mathbf{WK} , while \mathbf{S}_2 is a quotient (see [1] for details).

Although the algebra \mathbf{WK} allows to define the logic PWK (upon setting $\{1, \alpha\}$ as designated values), its relevance is a consequence also of the fact that it generates the variety \mathcal{IBSL} , [1, Corollary 31].

3. THE CATEGORIES OF STRONGLY INVERSE AND DIRECT SYSTEMS

The concepts that we are going to introduce in this section have been already treated before, though in a different setting and using a different language (see [15] and [14]). For our purposes we need to strengthen the concepts of inverse and direct system of a category, introducing the notions of strongly inverse and strongly direct systems. For sake of simplicity, we opt for presenting this topics following the trend in algebraic topology (see [7] for details).

Definition 3.1. Let \mathfrak{C} be an arbitrary category, a *strongly inverse system* in the category \mathfrak{C} is a tern $\mathcal{X} = \langle X_i, p_{ii'}, I \rangle$ such that

- (i) I is a join semilattice with lower bound;
- (ii) for each $i \in I$, X_i is an object in \mathfrak{C} ;
- (iii) $p_{ii'} : X_{i'} \rightarrow X_i$ is a morphism of \mathfrak{C} , for each pair $i \leq i'$, satisfying that p_{ii} is the identity in X_i and such that $i \leq i' \leq i''$ implies $p_{ii'} \circ p_{i'i''} = p_{ii''}$.

I is called the *index set* of the system \mathcal{X} , X_i are the *terms* and $p_{ii'}$ are referred to as *bonding morphisms* of \mathcal{X} . For convention, we indicate with \vee the semilattice operation on I , \leq the induced order and i_0 the lower bound in I .

The only difference making an inverse system a strongly inverse system is the requirement on the index set to be a semilattice with lower bound instead of a directed preorder.

Definition 3.2. Given two *strongly inverse systems* $\mathcal{X} = \langle X_i, p_{ii'}, I \rangle$ and $\mathcal{Y} = \langle Y_j, q_{jj'}, J \rangle$, a *morphism* between \mathcal{X} and \mathcal{Y} is a pair (φ, f_j) such that

- i) $\varphi : J \rightarrow I$ is a semilattice homomorphism;
- ii) for each $j \in J$, $f_j : X_{\varphi(j)} \rightarrow Y_j$ is a morphism in \mathfrak{C} , such that whenever $j \leq j'$, then the diagram in Fig.3 commutes.

$$\begin{array}{ccc}
 X_{\varphi(j)} & \xleftarrow{p_{\varphi(j)\varphi(j')}} & X_{\varphi(j')} \\
 f_j \downarrow & & \downarrow f_{j'} \\
 Y_j & \xleftarrow{q_{jj'}} & Y_{j'}
 \end{array}$$

FIGURE 1. The commuting diagram defining morphisms of strongly inverse systems

Notice that, for morphisms of strongly inverse systems, the assumption that $\varphi : J \rightarrow I$ is a (semilattice) homomorphism implies that whenever $j \leq j'$ then $\varphi(j) \leq \varphi(j')$. Given three strongly inverse systems $\mathcal{X} = \langle X_i, p_{ii'}, I \rangle$, $\mathcal{Y} = \langle Y_j, q_{jj'}, J \rangle$, $\mathcal{Z} = \langle Z_k, r_{kk'}, K \rangle$, the composition of morphisms is defined in the same way as for inverse systems.

Lemma 3.3. *The composition of morphisms between strongly inverse system is a morphism.*

Proof. Let $(\varphi, f_j) : \mathcal{X} \rightarrow \mathcal{Y}$, $(\psi, g_k) : \mathcal{Y} \rightarrow \mathcal{Z}$, then $(\chi, h_k) = (\psi, g_k)(\varphi, f_j) : \mathcal{X} \rightarrow \mathcal{Z}$ is $\chi = \varphi\psi$, $h_k = g_k f_{\chi(k)}$. χ is the composition of two (semilattice) homomorphisms, hence it is a semilattice homomorphism. The claim follows from the commutativity of the following diagram (we omitted the indexes for the maps p, q, r, f, g to make the notation less cumbersome)

$$\begin{array}{ccc}
X_{\chi(k)} & \xleftarrow{p} & X_{\chi(k')} \\
f \downarrow & & \downarrow f \\
Y_{\psi(k)} & \xleftarrow{q} & Y_{\psi(k')} \\
g \downarrow & & \downarrow g \\
Z_k & \xleftarrow{r} & Z_{k'}
\end{array}$$

□

Proposition 3.4. *Let \mathfrak{C} an arbitrary category. Then $\text{strong-inv-}\mathfrak{C}$ is the category whose objects are strongly inverse systems in \mathfrak{C} with morphisms as defined above.*

Proof. The composition of morphisms between systems is associative and the identity morphism is $(1_I, 1_i)$, where $1_I : I \rightarrow I$ is the identity homomorphism on I and $1_i : X_i \rightarrow X_i$ is the identity morphism in the category \mathfrak{C} . □

The category of *strongly direct systems* of a given category \mathcal{C} is obtained by reversing morphisms of $\text{strong-inv-}\mathcal{C}$ as follows:

Definition 3.5. Let \mathfrak{C} be an arbitrary category. A *strongly direct system* in \mathfrak{C} is a tern $\mathbb{X} = \langle X_i, p_{ii'}, I \rangle$ such that

- (i) I is a join semilattice with least element.
- (ii) X_i is an object in \mathfrak{C} , for each $i \in I$;
- (iii) $p_{ii'} : X_i \rightarrow X_{i'}$ is a morphism of \mathfrak{C} , for each pair $i \leq i'$, satisfying that p_{ii} is the identity in X_i and such that $i \leq i' \leq i''$ implies $p_{i'i''} \circ p_{ii'} = p_{ii''}$.

We call I , X_i , the index set and the terms of the direct system, respectively, while we refer to $p_{ii'}$ as *transition morphisms* to stress the crucial difference with respect to inverse systems.

A morphism between two strongly direct systems \mathbb{X} and \mathbb{Y} is a pair $(\varphi, f_i) : \mathbb{X} \rightarrow \mathbb{Y}$ s. t.

- i) $\varphi : I \rightarrow J$ is a semilattice homomorphism
- ii) $f_i : X_i \rightarrow Y_{\varphi(i)}$ is a morphism of \mathfrak{C} , making the following diagram commutative for each $i, i' \in I$, $i \leq i'$:

The composition of two morphisms is defined as $(f_i, \varphi)(g_j, \psi) = (h_i, \chi)$,

$$\chi = \psi\varphi, \quad h_i = g_{\varphi(i)}f_i : X_i \rightarrow Z_{\chi(i)}.$$

$$\begin{array}{ccc}
X_i & \xrightarrow{p_{ii'}} & X_{i'} \\
f_i \downarrow & & \downarrow f_{i'} \\
Y_{\varphi(i)} & \xrightarrow{q_{\varphi(i)\varphi(i')}} & Y_{\varphi(i')}
\end{array}$$

FIGURE 2. The commuting diagram defining morphisms of strongly direct systems

It is easily verified that the composition (h_i, χ) is a morphism and it is associative and that the element $(1_I, 1_i)$, where $1_I : I \rightarrow I$ is the identity map on I and $1_i : X_i \rightarrow X_i$ is the identity morphism in \mathfrak{C} , is the identity morphism between strongly direct systems. Therefore strongly direct systems form a category which we will call $\text{strong-dir-}\mathfrak{C}$.

Remark 3.6. If \mathfrak{C} and \mathfrak{D} are dual categories, then $\text{strong-dir-}\mathfrak{C}$ is the dual category of $\text{strong-inv-}\mathfrak{D}$.

4. THE CATEGORY OF INVOLUTIVE BISEMILATTICES AND ITS DUAL

Plonka introduced [8], [10] a construction to build algebras out of strongly direct systems of algebras².

Definition 4.1. If \mathbb{A} is a strongly direct system of algebras of a fixed type ν , then the *Plonka sum* over \mathbb{A} is the algebra $\mathcal{P}_l(\mathbb{A}) = \langle \bigsqcup_I A_i, g^{\mathcal{P}} \rangle$, whose universe is the disjoint union and the operations $g^{\mathcal{P}}$ are defined as follows: for every n -ary $g \in \nu$, and $a_1, \dots, a_n \in \bigsqcup_I A_i$, where $n \geq 1$ and $a_r \in A_{i_r}$, we set $j = i_1 \vee \dots \vee i_n$ and define

$$g^{\mathcal{P}}(a_1, \dots, a_n) = g^{\mathbf{A}_j}(\varphi_{i_1 j}(a_1), \dots, \varphi_{i_n j}(a_n)).$$

In case ν contains constants, then, for every constant $g \in \nu$, we define $g^{\mathcal{A}} = g^{\mathbf{A}_{i_0}}$.

Involutive bisemilattices, as well as bisemilattices, admits a representation in terms of Plonka sums.

Theorem 4.2 ([1, Thm. 46]).

- 1) If \mathbb{A} is a strongly direct system of Boolean algebras, then the $\mathcal{P}_l(\mathbb{A})$ is an involutive bisemilattice.
- 2) If \mathbf{B} is an involutive bisemilattice, then \mathbf{B} is isomorphic to the Plonka sum over a strongly direct system of Boolean algebras³.

²In the original terminology, Plonka called them *direct systems* and assumed the index set to be a semilattice. We slightly changed the name according to the previous section and the trend in algebraic topology.

³The form of the strongly direct system used in the Plonka sum representation is not needed for the purposes of this work. For more details, the reader could refer to [1].

The above result states that every involutive bisemilattice admits a unique representation as Plonka sum of Boolean algebras. We summarize here the categories we are dealing with

Category	Objects	Morphisms
\mathfrak{BA}	Boolean Algebras	Homomorphisms of B.A.
\mathfrak{IBSL}	Involutive bisemilattices	Homomorphisms of I.B.
strong-dir- \mathfrak{BA}	strongly direct systems of B.A.	Homomorphisms of s.d.s.
\mathfrak{SA}	Stone spaces	continuous maps
strong-inv- \mathfrak{SA}	strongly inverse s. of Stone sp.	Homomorphisms of s.i.s.

Theorem 4.2 states that the objects of the category \mathfrak{IBSL} are isomorphic to the objects of the category strong-dir- \mathfrak{BA} . Actually, they are also equivalent as categories (see Proposition 4.5).

In order to establish this, we need the following lemmata.

Lemma 4.3. *Let $\mathbb{A} = \langle A_i, p_{ii'}, I \rangle$ and $\mathbb{B} = \langle B_j, q_{jj'}, J \rangle$ be strongly direct systems of Boolean algebras and $h \in \text{Hom}(\mathcal{P}_l(\mathbb{A}), \mathcal{P}_l(\mathbb{B}))$, then for any $i \in I$ there exists a $j \in J$ such that*

- (1) $h(A_i) \subseteq B_j$
- (2) $h|_{A_i}$ is a Boolean homomorphism from A_i into B_j

Proof. (1) As first notice that, from the construction of Plonka sums, we have that for any $x \in A_i$, also $x' \in A_i$. Consequently, for any $h(x) \in B_j$, for a certain $j \in J$, then also $h(x)' \in B_j$. Let $a \in A_i$ for some $i \in I$, then there exists a $j \in J$ such that $h(a) \in B_j$. Therefore $h(0_{A_i}) = h(a \wedge a') = h(a) \wedge h(a') = h(a) \wedge h(a)' = 0_{B_j}$, where the last equality holds since $h(a)$ and $h(a)'$ belong to the same Boolean algebra B_j . Similarly, $h(1_{A_i}) = h(a \vee a') = h(a) \vee h(a') = h(a) \vee h(a)' = 1_{B_j}$.

We now have to prove that for any $a \in A_i$, with $a \neq 0_{A_i}$ we have that $h(a) \in B_j$. Suppose, by contradiction, that $a \in A_i$, and $h(a) \in B_k$, with $j \neq k$. Then $0_{B_j} = h(0_{A_i}) = h(a \wedge a') = h(a) \wedge h(a') = h(a) \wedge h(a)' = 0_{B_k}$, which is impossible, as, by construction $B_j \cap B_k = \emptyset$, hence, necessarily $h(A_i) \subseteq B_j$.

(2) follows from the fact that h preserves joins, meets and complements by definition and we already proved that $h(0_{A_i}) = 0_{B_j}$ and $h(1_{A_i}) = 1_{B_j}$. \square

Theorem 4.2 together with Lemma 4.3 state that \mathfrak{IBSL} -homomorphisms are nothing but homomorphisms between the correspondent (unique) Plonka sum representations. The statement of Lemma 4.3 can be exposed more precisely saying that there exists a map $\varphi : I \rightarrow J$ such that for every homomorphism $h : \mathcal{P}_l(\mathbb{A}) \rightarrow \mathcal{P}_l(\mathbb{B})$, $h(A_i) \subseteq B_{\varphi(i)}$. It is not difficult to prove that such map is actually a semilattice homomorphism.

Lemma 4.4. *Let $\mathbb{A} = \langle A_i, p_{ii'}, I \rangle$ and $\mathbb{B} = \langle B_j, q_{jj'}, J \rangle$ be strongly direct systems of Boolean algebras, $h \in \text{Hom}(\mathcal{P}_l(\mathbb{A}), \mathcal{P}_l(\mathbb{B}))$ and $\varphi_h : I \rightarrow J$ such that $h(A_i) \subseteq B_{\varphi_h(i)}$. Then φ_h is a semilattice homomorphism.*

Proof. Let $a_1 \in A_i$ and $a_2 \in A_{i'}$, with $i, i' \in I$; by definition of $\mathcal{P}_l(\mathbb{A})$, $a_1 \wedge a_2 \in A_{i \vee i'}$ and $h(a_1) \in B_{\varphi_h(i)}$, $h(a_2) \in B_{\varphi_h(i')}$, then $h(a_1 \wedge a_2) = h(a_1) \wedge h(a_2) \in B_{\varphi_h(i) \vee \varphi_h(i')}$. But since $h(a_1 \wedge a_2) \in B_{\varphi_h(i \vee i')}$, then necessarily $\varphi_h(i \vee i') = \varphi_h(i) \vee \varphi_h(i')$, i.e. φ_h is a semilattice homomorphism. \square

Proposition 4.5. *The category \mathfrak{IBSL} is isomorphic to strong-dir- \mathfrak{BA} .*

Proof. By Theorem 4.2, the objects in the two categories are isomorphic. On the other hand, by Lemmata 4.3 and 4.4 morphisms coincide in the two categories. \square

Recall that a *Stone space* is topological space which is compact, Hausdorff and totally disconnected. Stone spaces can be viewed as a category, which we refer to as \mathfrak{SA} with continuous maps as morphisms.

It is well known that the category of Stone spaces is the dual of the category of Boolean algebras [16]. Proposition 4.5 and Remark 3.6 yields the following first characterization of the dual category of \mathfrak{IBSL} .

Theorem 4.6. *The category strong-inv- \mathfrak{SA} is the dual of \mathfrak{IBSL} .*

Theorem 4.6 gives a description of the dual category of involutive bisemilattices in terms of Stone spaces, i.e. the natural dual category of Boolean algebras, objects coming into play due to the representation Theorem 4.2.

The above theorem together with Theorem 4.2 should be compared with the following theorem due to Haimo [3], where direct limits are considered instead of Płonka sums. In the following statement, \lim_{\rightarrow} , \lim_{\leftarrow} denote the direct and inverse limit, respectively.

Theorem 4.7 ([3], Th. 9). *Let $\{\mathbf{A}_i\}$ be a direct system of Boolean algebras and $\{\mathbf{A}_i^*\}$ the corresponding family of Stone spaces. Then*

$$(\lim_{\rightarrow} \mathbf{A}_i)^* \cong \lim_{\leftarrow} \mathbf{A}_i^*.$$

In Theorem 4.18 (see below) we will give a *concrete* topological description of the dual space of an involutive bisemilattices based on Gierz and Romanowska duality (see Theorem 2.6), where, instead of GR spaces, we use GR spaces with involution.

Definition 4.8. A *GR space with involution* is a GR space \mathbf{G} with a continous map $\neg : G \rightarrow G$ such that for any $a \in G$:

G1. $\neg(\neg a) = a$

G2. $\neg(a * b) = \neg a * \neg b$

G3. if $a \leq b$ then $\neg b \sqsubseteq \neg a$

G4. $\neg c_0 = c_1$, $\neg c_1 = c_0$ and $\neg c_\alpha = c_\alpha$

G5. The space $\text{Hom}_{\text{GR}}(\mathbf{A}, \mathbf{3})$ (see Section 2) equipped with natural involution \neg , i.e. $\neg\varphi(a) = (\varphi(\neg a))'$ satisfies $\varphi \cdot (\neg\varphi + \psi) = \psi \cdot \varphi$, where operations are defined pointwise;

G6. there exist $\varphi_0, \varphi_1 \in \text{Hom}_{\text{GR}}(\mathbf{A}, \mathbf{3})$ such that $\neg\varphi_0 = \varphi_1$ and $\varphi + \varphi_0 = \varphi$, for each $\varphi \in \text{Hom}_{\text{GR}}(\mathbf{A}, \mathbf{3})$.

Example 4.9. **WK** equipped with discrete topology is the canonical example of GR space with involution.

Definition 4.10. \mathfrak{GR} is the category whose objects are GR spaces with involution and whose morphisms are GR-morphisms preserving involution.

Given a GR space with involution \mathbf{G} , we can consider its GR space reduct (simply its involution free reduct), say \mathbf{A} , which can be associated to the dual distributive bisemilattice $\hat{\mathbf{A}} = \text{Hom}_{\text{GR}}(\mathbf{A}, \mathbf{3})$. Aiming at turning it into an involutive bisemilattice, we define an involution on $\hat{\mathbf{A}}$ as follows:

$$\neg\Phi(a) = (\Phi(\neg a))',$$

for each $\Phi \in \hat{\mathbf{A}}$ and $a \in G$, where \neg and $'$ are the involutions of \mathbf{G} and **WK**, respectively.

Lemma 4.11. *If $\Phi \in \hat{\mathbf{A}}$ then $\neg\Phi \in \hat{\mathbf{A}}$.*

Proof. Assuming that Φ is a morphism of GR spaces, we have to verify that also $\neg\Phi$ is, i.e. that it is a continuous map, preserving operation $*$, constants and the order \leq . Observe that $\neg\Phi$ is continuous as it is the composition of continuous maps.

Concerning operations and constants, we have:

$$\neg\Phi(a * b) = (\Phi(\neg(a * b)))' = (\Phi(\neg a * \neg b))' = (\Phi(\neg a) * \Phi(\neg b))' = (\Phi(\neg a))' * (\Phi(\neg b))' = \neg\Phi(a) * \neg\Phi(b).$$

$$\neg\Phi(c_0) = (\Phi(\neg c_0))' = (\Phi(c_1))' = 1' = 0.$$

$$\text{Similarly, } \neg\Phi(c_1) = (\Phi(\neg c_1))' = (\Phi(c_0))' = 0' = 1 \text{ and } \neg\Phi(c_\alpha) = (\Phi(\neg c_\alpha))' = (\Phi(c_\alpha))' = \alpha' = \alpha.$$

As for the order, let $a \leq b$, but then $\neg b \sqsubseteq \neg a$. Since Φ preserve both the orders, $\Phi(\neg b) \leq_+ \Phi(\neg a)$, thus $(\Phi(\neg a))' \leq (\Phi(\neg b))'$, i.e. $\neg\Phi(a) \leq \neg\Phi(b)$. \square

Proposition 4.12. $\hat{\mathbf{G}} = \langle \hat{\mathbf{A}}, \neg \rangle$ is an involutive bisemilattice.

Proof. We have to check that conditions **I1** to **I8** of Definition 2.7 hold for $\hat{\mathbf{G}}$. Clearly, **I1**, **I2** and **I3** hold as $\hat{\mathbf{A}}$ is a distributive bisemilattice, while **I6**, **I7** and **I8** hold by definition. For the remaining ones, let $\varphi \in \hat{\mathbf{A}}$ and $a \in A$.

$$\mathbf{I4.} \quad \neg(\neg\varphi(a)) = \neg\varphi(\neg(a))' = \varphi(\neg\neg a)'' = \varphi(a).$$

$$\mathbf{I5.} \quad \neg(\varphi + \psi)(a) = (\varphi + \psi(\neg a))' = (\varphi(\neg a) + \psi(\neg a))' = (' \varphi(\neg a))' \cdot (\psi(\neg a))' = \neg\varphi(a) \cdot \neg\psi(a). \quad \square$$

Proposition 4.13. $\mathbf{G} \cong \hat{\hat{\mathbf{G}}}$.

Proof. We make good use of the duality established in [2], from which it follows $\mathbf{A} \cong \hat{\hat{\mathbf{A}}}$, where \mathbf{A} is the GR space reduct of \mathbf{G} . To prove our claim we only have to prove that the isomorphism, given by (2), $\delta_A(x)(\varphi) = \varphi(x)$, for $x \in A$ and $\varphi \in \hat{\mathbf{A}}$, preserve the involution. This is easily checked, indeed

$$(\neg\delta_A(x))(\varphi) = (\delta_A(x)(\neg\varphi))' = (\neg\varphi(x))' = (\varphi(\neg x))'' = \varphi(\neg x).$$

□

Given an arbitrary involutive bisemilattice \mathbf{I} , we consider its bisemilattice reduct $\mathbf{S} = \langle I, +, \cdot \rangle$, which is distributive [1, Proposition 20], and therefore can be associated to its natural dual GR space, $\hat{\mathbf{S}} = \text{Hom}_b(\mathbf{S}, \mathbf{3})$ (see Section 2). The bisemilattice $\mathbf{3}$ turns into \mathbf{WK} just by adding the usual involution and the constants 0, 1, so it makes sense to define an involution on $\hat{\mathbf{S}}$ as:

$$\neg\varphi(x) = (\varphi(x'))',$$

for any $\varphi \in \hat{\mathbf{S}}$ and $x \in S$. We prove that $\hat{\mathbf{S}}$ is closed with respect to the above defined involution.

Lemma 4.14. *If $\varphi \in \hat{\mathbf{S}}$ then $\neg\varphi \in \hat{\mathbf{S}}$.*

Proof. Suppose that $\varphi \in \hat{\mathbf{S}}$, i.e. it is a map preserving sum and multiplication. It suffices to verify that also $\neg\varphi$ preserves the two operations. $\neg\varphi(x + y) = (\varphi(x + y))' = (\varphi(x') \cdot \varphi(y'))' = (\varphi(x') \cdot \varphi(y'))' = (\varphi(x'))' + (\varphi(y'))' = \neg\varphi(x) + \neg\varphi(y)$.

For multiplication the proof runs analogously. □

Remark 4.15. The role of the dual space can not be played by $\text{Hom}_i(\mathbf{I}, \mathbf{WK})$, the space of homomorphisms of involutive bisemilattices (namely those maps preserving also involution) with $\varphi'(x) = \varphi(x)'$. Indeed $\text{Hom}_i(\mathbf{I}, \mathbf{WK})$ is not closed under such involution: $\varphi'(x + y) = (\varphi(x + y))' = (\varphi(x) + \varphi(y))' = \varphi'(x) \cdot \varphi'(y)$, which is in general different from $\varphi'(x) + \varphi'(y)$.

The above Remark highlights the fact that the duality presented in this paper cannot be constructed following the usual prescription of “schizofrenic” objects and homo functors. For example, in the case of Stone duality, the two element boolean algebra $\mathbf{2} = \{0, 1\}$ is the schizofrenic object as the duality is constructed looking at it once as a Boolean algebra and once as a Stone space. The same idea is used also in the duality for distributive bisemilattices [2], with $\mathbf{3}$ as schizofrenic object. In our case, the object \mathbf{WK} is still chosen to belong to the target category and its dual and in this sense the duality is natural.

Proposition 4.16. $\hat{\mathbf{I}} = \langle \hat{\mathbf{S}}, \neg \rangle$ is a GR space with involution.

Proof. By [2], we have that $\hat{\mathbf{S}}$ is a GR space, thus we only have to check that \neg has the required properties. Let $\varphi, \psi \in \hat{\mathbf{S}}$ and $x \in S$; properties **G1** – **G4** can be easily verified as follows:

$$\neg(\neg\varphi(x)) = \neg(\varphi(x'))' = (\varphi(x''))'' = \varphi(x).$$

$$\neg(\varphi * \psi)(x) = (\varphi * \psi(x'))' = (\varphi(x') * \psi(x'))' = (\varphi(x'))' * (\psi(x'))' = \neg\varphi(x) * \neg\psi(x).$$

Let $\varphi \leq \psi$, i.e. $\varphi(x) \leq \psi(x)$ for each $x \in S$. In particular $\varphi(x') \leq \psi(x')$, thus $(\psi(x'))' \leq_+ (\varphi(x'))'$, i.e. $\neg\psi \sqsubseteq \neg\varphi$.

Let φ_0, φ_1 and φ_α the constant homomorphisms (of bisemilattices) on 0, 1 and α , respectively. $\neg\varphi_0(x) = (\varphi_0(x'))' = 0' = 1 = \varphi_1(x)$; $\neg\varphi_1(x) = (\varphi_1(x'))' = 1' = 0 = \varphi_0(x)$; $\neg\varphi_\alpha(x) = (\varphi_\alpha(x'))' = \alpha' = \alpha = \varphi_\alpha(x)$.

In order to prove **G5** and **G6**, it is enough to show that $\mathbf{I} \cong \hat{\hat{\mathbf{I}}}$. Recall that the bisemilattice reduct \mathbf{S} of \mathbf{I} is isomorphic to $\hat{\hat{\mathbf{S}}}$ under the isomorphism given by (1), namely $\varepsilon_s(x)(\varphi) = \varphi(x)$, for every $\varphi \in \hat{S}$ and $x \in S$. The map ε_s is obviously a homomorphism of bisemilattices and a bijection from $\mathbf{I} \setminus \{0, 1\}$ to $\hat{\hat{\mathbf{I}}} \setminus \{\Phi_0, \Phi_1\}$, where by Φ_0, Φ_1 we indicate the constants in $\hat{\hat{\mathbf{I}}}$. This map can be extended to a bijection from \mathbf{I} to $\hat{\hat{\mathbf{I}}}$, by setting $\varepsilon_s(0) = \Phi_0$ and $\varepsilon_s(1) = \Phi_1$. We have to prove that Φ_0 and Φ_1 indeed play the role of the constants in $\hat{\hat{\mathbf{I}}}$ and that ε_s also preserves involution. We start with the latter task:

$$(\neg\varepsilon_s(x))(\varphi) = (\varepsilon_s(x)(\neg\varphi))' = (\neg\varphi(x))' = (\varphi(x'))'' = \varphi(x').$$

Regarding the constants, we only need to prove that $\neg\Phi_0 = \Phi_1$ and $\Psi + \Phi_0 = \Psi$, for each $\Psi \in \hat{\hat{\mathbf{I}}}$. Indeed, for any $\varphi \in \hat{\mathbf{I}}$, one has:

$$\neg\Phi_0(\varphi) = \neg\varepsilon_s(0)(\varphi) = \varphi(0') = \varphi(1) = \varepsilon_s(1)(\varphi) = \Phi_1(\varphi).$$

Finally, due to the surjectivity of ε_s , for any $\Psi \in \hat{\hat{\mathbf{I}}}$, there exists $x \in I$ such that $\Psi = \varepsilon_s(x)$. Therefore $\Psi(\varphi) = \varepsilon_s(x)(\varphi) = \varepsilon_s(x+0)(\varphi) = \varphi(x+0) = \varphi(x) + \varphi(0) = \varepsilon_s(x)(\varphi) + \varepsilon_s(0)(\varphi) = (\Psi + \Phi_0)(\varphi)$ and we are done. \square

In order to prove Theorem 4.18 we are only left with proving that the functors $\text{Hom}_{\mathbf{b}}(-, \mathbf{WK}) : \mathfrak{IBSL} \rightarrow \mathfrak{IGN}$ and $\text{Hom}_{\text{GR}}(-, \mathbf{WK}) : \mathfrak{IGN} \rightarrow \mathfrak{IBSL}$ are contravariant (we consider just the first functor as for the other the proof runs analogously).

Proposition 4.17. *Let $f : \mathbf{I} \rightarrow \mathbf{L}$ be a morphism of \mathfrak{IBSL} , then it induces a morphism of \mathfrak{IGN} $f^* : \hat{\mathbf{L}} \rightarrow \hat{\mathbf{I}}$, where $\hat{\mathbf{L}}, \hat{\mathbf{I}}$ are the dual spaces of \mathbf{L} and \mathbf{I} , respectively.*

Proof. f^* is defined in the usual way, i.e. $f^*(\hat{j})(i) = \hat{j}(f(i))$, for each $i \in \mathbf{I}$ and $\hat{j} \in \hat{\mathbf{J}}$. It suffices to prove that f^* preserves involution, namely $f^*(\neg\hat{j}) = \neg f^*(\hat{j})$, for all $\hat{j} \in \hat{\mathbf{J}}$:

$$(\neg f^*(\hat{j}))(i) = \neg\hat{j}(f(i)) = f^*(\neg\hat{j})(i),$$

\square

Surprisingly enough, we have established that strongly inverse systems of Stone spaces are nothing but GR spaces with involution.

Theorem 4.18. *The of GR spaces with involution is the dual of the category of involutive bisemilattices.*

Corollary 4.19. *The category strong-inv- $\mathfrak{S}\mathfrak{A}$ is equivalent to the category of GR spaces with involution.*

Theorem 4.18 highlights an interesting as well as unexpected topological properties of Stone spaces. Indeed the category of (strongly) inverse systems of Stone spaces which deals with a possibly infinite family of them can be described by a specific class of topological spaces, namely GR spaces with involution.

5. FINAL COMMENTS AND REMARKS

It is natural to wonder whether the result in Theorem 4.18 may be extended to other algebraic categories admitting topological duals such as bisemilattices and GR spaces. Indeed, recall that bisemilattices are Plonka sums of distributive lattices, according to the following

Theorem 5.1. [9, Th. 3] *An algebra \mathbf{B} is a bisemilattice iff it is the Plonka sum over a strongly direct system of distributive lattices.*

A *Priestley space* is an ordered topological space, i.e. a set X equipped with a partial order \leq and a topology τ , such that $\langle X, \tau \rangle$ is compact and, for $x \not\leq y$ there exists a clopen up-set U such that $x \in U$ and $y \notin U$. The category of Priestley spaces, \mathfrak{PS} , is the category whose objects are Priestley spaces and morphisms are continuous maps preserving the ordering.

The category of Priestley spaces is the dual of the category of distributive lattices [12], [13].

Let us call \mathfrak{BSL} the category of bisemilattices (objects are bisemilattices, morphisms homomorphisms of bisemilattices). It follows from our analysis and Theorem 5.1 that the objects in \mathfrak{BSL} are the same as in strong-dir- \mathfrak{DL} , where \mathfrak{DL} stands for the category of distributive lattices. We claim that the two categories of \mathfrak{BSL} and strong-dir- \mathfrak{DL} coincide. We show that by using the same strategy applied in Section 4.

Lemma 5.2. *Let \mathbf{L} and \mathbf{M} be two bisemilattices, the Plonka sums over the strongly direct systems of distributive lattices $\mathbb{L} = \langle L_i, \varphi_{i,i'}, I \rangle$ and $\mathbb{M} = \langle M_j, \varphi_{j,j'}, J \rangle$, and let $h : \mathbf{L} \rightarrow \mathbf{M}$ be a homomorphism. Then, for any $i \in I$, there exists a $j \in J$ such that $h(L_i) \subseteq M_j$.*

Moreover, there exists a semilattice homomorphism $\varphi : I \rightarrow J$, for every homomorphism $h : \mathcal{P}_l(\mathbb{L}) \rightarrow \mathcal{P}_l(\mathbb{M})$, $h(A_i) \subseteq B_{\varphi(i)}$.

Proof. Let $a, b \in L_i$: we claim that $h(a), h(b) \in M_j$, for some $j \in J$. Two cases may arise: either a, b are comparable with respect to the order \leq of L_i or they are not. Suppose a and b are comparable: let $a \leq b$ and suppose that $h(a) \in M_j$, $h(b) \in M_{j'}$ with $j \neq j'$. Then, $h(a) = h(a \wedge b) = h(a) \wedge h(b) \in M_{j \vee j'}$ (by definition of operations in the Plonka sum), therefore $j = j \vee j'$. On the other hand, $h(b) = h(a \vee b) = h(a) \vee h(b) \in M_{j \vee j'}$. Thus $j = j'$.

The case of $b < a$ can be proved analogously.

Suppose now that a is not comparable with b , namely $a \not\leq b$ and $b \not\leq a$. Clearly $a \wedge b \leq a \vee b$, hence, reasoning as above, $h(a \vee b)$ and $h(a \wedge b)$ will belong to the same M_j for some $j \in J$. Now, both a and b are comparable with $a \wedge b$ and $a \vee b$, hence necessarily $h(a) \in M_j$ and $h(b) \in M_j$. Therefore $h(L_i) \in M_j$.

The proof of the second statement runs analogously as for Lemma 4.4. \square

As consequence of Theorem 5.1 and Lemma 5.2 we get

Proposition 5.3. *The category \mathfrak{BSL} is equivalent to strong-dir- \mathfrak{DL} .*

Using Priestley duality and Remark 3.6 we have

Theorem 5.4. *The category strong-inv- \mathfrak{PS} is the dual of \mathfrak{BSL} .*

As the category of GR spaces is the dual category of \mathfrak{BSL} (see Theorem 2.6), this means that strong-inv- \mathfrak{PS} are equivalent to a single class of spaces, namely

Corollary 5.5. *The category strong-inv- \mathfrak{PS} is equivalent to the category of GR spaces.*

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